Q1

i) $3\sec^2 3x + (-4x)e^{7-2x^2}$

$$= \boxed{3\sec^2 3x - 4xe^{7-2x^2}}$$

ii) Let $u = x^2+2x-8$ $\quad v = \cos(3-x)$

$\frac{du}{dx} = 2x+2 = 2(x+1)$ $\quad \frac{dv}{dx} = -1(-\sin(3-x)) = \sin(3-x)$

$$\frac{d(uv)}{dx} = \boxed{(x^2+2x-8)\sin(3-x) + 2(x+1)\cos(3-x)}$$

Alternatively, $= -(x^2+2x-8)\sin(x-3) + 2(x+1)\cos(x-3)$

since $\cos(-x) = \cos x \quad \therefore \cos(3-x) = \cos(x-3)$

$\sin(-x) = -\sin x \quad \therefore \sin(3-x) = -\sin(x-3)$

CHAIN RULE

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

PRODUCT RULE

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

QUOTIENT RULE

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

iii) Let $u = \ln 7x$ $\quad v = \sin(x^2+5)$

$\frac{du}{dx} = \frac{7}{7x} = \frac{1}{x}$ $\quad \frac{dv}{dx} = 2x\cos(x^2+5)$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{\frac{\sin(x^2+5)}{x} - 2x\ln 7x\cos(x^2+5)}{\sin^2(x^2+5)}$$

$$= \boxed{\frac{\sin(x^2+5) - 2x^2\ln 7x\cos(x^2+5)}{x\sin^2(x^2+5)}}$$

iv) $\frac{d(\cos 4x)}{dx} = -4\sin 4x$

$$\frac{d(\sqrt{\cos 4x})}{dx} = (-4\sin 4x)\tfrac{1}{2}(\cos 4x)^{-\frac{1}{2}}$$

$$= \boxed{-2\sin 4x(\cos 4x)^{-\frac{1}{2}}}$$

Q2

$y = \left(e^{\ln 3}\right)^{x} + \left(e^{(\ln 2)}\right)^{-x}$ $a = e^{\ln a}$

$y = e^{x\ln 3} + e^{-x\ln 2}$

Chain rule

$\frac{dy}{dx} = (\ln 3)e^{x\ln 3} - (\ln 2)e^{-x\ln 2}$

At $x = 1$

$\frac{dy}{dx} = (\ln 3)e^{1\times\ln 3} - (\ln 2)\boxed{e^{-1\times\ln 2}}$

$e^{-\ln 2} = \frac{1}{e^{\ln 2}} = \frac{1}{2}$

$= 3\ln 3 - \frac{1}{2}\ln 2$

$m_{normal} = \frac{-1}{\frac{dy}{dx}} = \frac{-1}{3\ln 3 - \frac{1}{2}\ln 2} \xrightarrow{\times\frac{-2}{-2}} = \boxed{\frac{2}{\ln 2 - 6\ln 3}}$

Q3

Start by using the chain rule to differenciate the inner function $\left(\ln\frac{1}{x}\right)$, and then work outwards.

$$\frac{d\left(\ln\frac{1}{x}\right)}{dx} = \left(-x^{-2}\right)\frac{1}{\frac{1}{x}} = -x^{-1}$$

$$\frac{d\left(\cos\left(\ln\frac{1}{x}\right)\right)}{dx} = \left(-x^{-1}\right)\left(-\sin\left(\ln\frac{1}{x}\right)\right)$$

$$= \frac{\sin\left(\ln\frac{1}{x}\right)}{x}$$

$$\frac{d\left(\sin\left(\cos\left(\ln\frac{1}{x}\right)\right)\right)}{dx} = \frac{\sin\left(\ln\frac{1}{x}\right)}{x}\cos\left(\cos\left(\ln\frac{1}{x}\right)\right)$$

$$\boxed{f'(x) = \frac{\sin\left(\ln\frac{1}{x}\right)\cos\left(\cos\left(\ln\frac{1}{x}\right)\right)}{x}}$$

Q4a

a) $y = \left(e^{\ln 4}\right)^{-x^4} = e^{-x^4 \ln 4}$ $\qquad a = e^{\ln a}$

Chain rule

$$\frac{dy}{dx} = \frac{d(-x^4 \ln 4)}{dx} e^{-x^4 \ln 4}$$

$$= -\boxed{4}(\ln 4)x^3 \boxed{e^{-x^4 \ln 4}}$$

$$4e^{-x^4 \ln 4} = 4e^{\ln 4^{-x^4}} = 4 \times 4^{-x^4} = 4^{1-x^4}$$

$$\therefore \boxed{\frac{dy}{dx} = -(\ln 4)x^3 4^{1-x^4}}$$

Q4b

b) $\quad y - y_1 = m(x - x_1)$

$y_1 \to \frac{1}{4}$, $\quad x_1 \to 1$

When $x = 1$,

$$m = \frac{dy}{dx} = -(\ln 4)(1)^3 4^{1-1^4} = -\ln 4$$

$$y - \frac{1}{4} = -\ln 4(x - 1)$$

$$\boxed{y = -(\ln 4)x + \left(\ln 4 + \frac{1}{4}\right)}$$

Q5a

Differentiate with respect to x, simplifying your answers where possible:

(a) $\underbrace{(5+\sin^2 3x)}_{u}\underbrace{e^{x^2-3x+2}}_{v}$ [3]

(b) $3^{\sqrt{x}}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$ [3]

Use the product rule!

$u = 5+\sin^2 3x$

$\;\; = 5+\frac{1}{2}(1-\cos 6x))$ double angle formula

$u' = \frac{1}{2}(6\sin 6x)$

$v = e^{x^2-3x+2}$

$v' = (2x-3)e^{x^2-3x+2}$

a) $\frac{d(uv)}{dx} = uv' + vu'$

$= (5+\sin^2 3x)(2x-3)e^{x^2-3x+2} + e^{x^2-3x+2}(3\sin 6x)$

$$\boxed{\frac{d(uv)}{dx} = \left(e^{x^2-3x+2}\right)\left((5+\sin^2 3x)(2x-3)+3\sin 6x\right)}$$

Q5b

Differentiate with respect to x, simplifying your answers where possible:

(a) $(5+\sin^2 3x)e^{x^2-3x+2}$ [3]

(b) $\underbrace{3^{\sqrt{x}}}_{u}\underbrace{\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)}_{v}$ [3]

Product rule

$u = 3^{x^{\frac{1}{2}}} = e^{\ln 3^{x^{\frac{1}{2}}}} = e^{x^{\frac{1}{2}}\ln 3}$ $\qquad a = e^{\ln a}$

$u' = \frac{1}{2}x^{-\frac{1}{2}}(\ln 3)e^{x^{\frac{1}{2}}\ln 3} = \frac{1}{2}x^{-\frac{1}{2}}(\ln 3)3^{x^{\frac{1}{2}}}$

$v = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

$v' = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$

b) $\frac{d(uv)}{dx} = uv' + vu'$

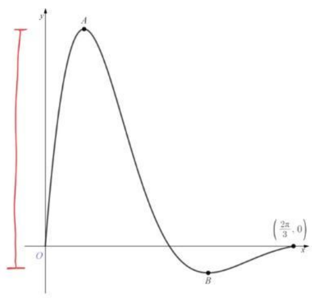
$= 3^{x^{\frac{1}{2}}}\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}\right) + \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\frac{1}{2}x^{-\frac{1}{2}}(\ln 3)3^{x^{\frac{1}{2}}}$

$= \frac{3^{x^{\frac{1}{2}}}}{2}\left(x^{-\frac{1}{2}} + x^{-\frac{3}{2}} + (\ln 3)x^{-\frac{1}{2}}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\right)$

$$\boxed{\frac{d(uv)}{dx} = \frac{3^{x^{\frac{1}{2}}}}{2}\left(x^{-\frac{1}{2}} + x^{-\frac{3}{2}} + \ln 3 - (\ln 3)x^{-1}\right)}$$

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is the function defined by

$$f(x) = \frac{\sin 3x}{e^{2x-3}}, = \frac{u}{v} \quad 0 \le x \le \frac{2\pi}{3}$$



The points A and B are maximum and minimum points, respectively.

Find the range of $f(x)$, giving your answer correct to 3 decimal places. [6]

$y_A \le f(x) \le y_B$ within the given range

$f'(x) = 0$ at stationary points

$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vu' - uv'}{v^2}$ quotient rule

$u = \sin 3x \qquad v = e^{2x-3}$

$u' = 3\cos 3x \qquad v' = 2e^{2x-3}$

$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{\cancel{e^{2x-3}}(3\cos 3x) - (\sin 3x)(2\cancel{e^{2x-3}})}{\cancel{(e^{2x-3})^2}} = 0$

$3\cos 3x - 2\sin 3x = 0$

$3\cos 3x = 2\sin 3x$

$\frac{3}{2} = \tan 3x$

$x = \frac{\tan^{-1}\left(\frac{3}{2}\right)}{3}, \; \frac{\tan^{-1}\left(\frac{3}{2}\right)+\pi}{3}$

When $x = \frac{\tan^{-1}\left(\frac{3}{2}\right)}{3}$ $\quad f(x) = 8.679\ldots \rightarrow$ point A

When $x = \frac{\tan^{-1}\left(\frac{3}{2}\right)+\pi}{3}$ $\quad f(x) = -1.068\ldots \rightarrow$ point B

Range $= f(x_A) - f(x_B) = 8.679\ldots - (-1.068\ldots)$

$\boxed{\text{Range} = 9.748}$ (3dp)

Q7

When $x = \frac{\pi\sqrt{2}}{4}$, the arguement in all the functions is $\frac{\pi}{2}$.

$$u_1 = \sin(x\sqrt{2}) = \sin\frac{\pi}{2} = 1$$
$$u_2 = \frac{d(\sin(x\sqrt{2}))}{dx} = \sqrt{2}\cos(x\sqrt{2}) = \sqrt{2}\cos\frac{\pi}{2} = 0$$
$$u_3 = -2\sin(x\sqrt{2}) = -2\sin\frac{\pi}{2} = -2$$
$$u_4 = -2\sqrt{2}\cos(x\sqrt{2}) = -2\sqrt{2}\cos\frac{\pi}{2} = 0$$
$$u_5 = 4\sin(x\sqrt{2}) = 4\sin\frac{\pi}{2} = 4$$

$\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$

So all the sin terms (21 of them) contain 1, and all the cos terms (20 of them) are 0.

Therefore the sum is:

$$1 - 2 + 4 - 8 + \ldots$$

$n = 21$ $\quad a = 1$ $\quad r = -2$ (geometic series)

$$S_n = \frac{a(1-r^n)}{1-r} \qquad S_{21} = \frac{1(1-(-2)^{21})}{1-(-2)} = 699051$$

$$\boxed{f_{41}\left(\frac{\pi\sqrt{2}}{4}\right) = 699051}$$

Q8

Use calculus to find the coordinates of the stationary points of the curve

$$y = \frac{x}{3} - \tan^{-1}\left(\frac{2x}{3}\right)$$

and determine whether each one is a maximum or a minimum. The coordinates should be given as exact values. [6]

$$\boxed{\frac{dy}{dx} = \frac{1}{3} - \left(\frac{2}{3}\right)\frac{1}{1+\left(\frac{2x}{3}\right)^2}} = 0$$

$$\frac{dy}{dx} = \frac{1}{3} - \frac{2}{3}\left(1+\frac{4x^2}{9}\right)^{-1}$$

$$\frac{1}{\cancel{3}} = \frac{2}{\cancel{3}\left(1+\frac{4x^2}{9}\right)}$$

$$1 + \frac{4x^2}{9} = 2$$

$$\frac{4x^2}{9} = 1$$

$$x^2 = \frac{9}{4}$$

$$x = \pm\frac{3}{2}$$

When $x = \frac{3}{2}$, $\quad y = \left(\frac{3}{2}\right)\frac{1}{3} - \tan^{-1}\left(\frac{2}{3}\left(\frac{3}{2}\right)\right) = \frac{1}{2} - \frac{\pi}{4}$

$x = -\frac{3}{2}$, $\quad y = \left(-\frac{3}{2}\right)\frac{1}{3} - \tan^{-1}\left(\frac{2}{3}\left(-\frac{3}{2}\right)\right) = -\frac{1}{2} + \frac{\pi}{4}$

Classify SPs

$$\frac{d^2y}{dx^2} = -\frac{2}{3}\left(\frac{8x}{9}\right)(-1)\left(1+\frac{4x^2}{9}\right)^{-2} = \frac{16x}{27}\underbrace{\left(1+\frac{4x^2}{9}\right)^{-2}}$$

This is +ve for any value of x. So the sign (+/-) of $\frac{d^2y}{dx^2}$ matches the sign of the x-coordinate!

When $x = \frac{3}{2}$, $\quad \frac{d^2y}{dx^2} > 0 \quad \therefore$ min point at $\left(\frac{3}{2}, \frac{1}{2} - \frac{\pi}{4}\right)$

$x = -\frac{3}{2}$, $\quad \frac{d^2y}{dx^2} < 0 \quad \therefore$ max point at $\left(-\frac{3}{2}, -\frac{1}{2} + \frac{\pi}{4}\right)$